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On the world function of the Gödel metric

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Abstract. The Hamilton–Jacobi equations obeyed by the world function Ω of a V_4 are two simultaneous equations in eight independent variables. We utilise the symmetries possessed by the Gödel metric to reduce these equations to a single equation in only two variables. Ω may then be exhibited in a parametric form involving only a single parameter. If no such parameters are to be admitted one has to have recourse to power series. Accordingly, after some remarks on analyticity, initial terms of various power series for Ω are calculated and certain of their features briefly discussed.

1. Introduction

The world function $\Omega(x, x')$ (Synge 1960) is a two-point scalar, loosely defined to be half the square of the geodesic distance between x and x'. Since it is effectively a characteristic function associated with the geodesics it contains all the integral information about them from the Hamiltonian viewpoint. It is the fundamental integral quantity which can be defined upon an arbitrary Riemannian manifold and, as one might expect, is of prime importance in obtaining the Green functions for wave equations on curved backgrounds (e.g. Hadamard 1952, Friedlander 1975). Again, DeWitt (1975) in his discussion of quantum field theory in curved space–time derives expressions for the Feynman propagator of massive scalar particles solely in terms of Ω and its concomitants.

One would naturally like to be able to calculate Ω in closed form. Hitherto this has been done only in certain special cases, distinguished by their symmetries (Buchdahl 1972, Warner 1978). In view of the high degree of symmetry of the Gödel metric one might expect its world function also to be obtainable in closed form. In fact it is not, although it is possible to exhibit Ω in a form which is, so to speak, only one step removed from being closed, namely in parametric form involving only a single parameter. If the presence of such a parameter is disallowed one has to be content with some form of approximation. A substantial part of this paper is therefore devoted to the expansion of Ω in power series, the approach taken being similar to that of Buchdahl and Warner (1979).

We take the metric in the usual form (Hawking and Ellis 1973)

$$ds^2 = -dt^2 + dx^2 - \tfrac{1}{2}\exp(2\sqrt{2}\omega x)\, dy^2 - 2\exp(\sqrt{2}\omega x)\, dt\, dy + dz^2. \tag{1.1}$$

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Observe that this metric is a direct sum of the two metrics

$$ds_1^2 = -dt^2 + dx^2 - \tfrac{1}{2}\exp(2\sqrt{2}\omega x)\, dy^2 - 2\exp(\sqrt{2}\omega x)\, dt\, dy \tag{1.2}$$

and

$$ds_2^2 = dz^2. \tag{1.3}$$

Hence the Gödel universe is the product of a V_3 and R^1. It will be shown in § 2 that in effect we therefore have to calculate only the world function Ω_1 of the metric (1.2). This metric has four Killing vector fields, and in § 3 we show that as a consequence the six variables upon which Ω_1 depends can only appear in two combinations, denoted by p and q. The problem is then reduced to solving a single partial differential equation in p and q. An equation very simply related to it is separable and it therefore becomes possible to find a parametric form of the world function involving only one parameter. The details are set out in § 4. In § 5, after showing that Ω_1 is an analytic function of the variables p and q (for p and q sufficiently small), early terms of the power series for Ω_1 are determined. In the two succeeding sections we investigate the series for Ω_1 in ascending powers of p on the one hand or of q on the other, with coefficients which are functions of q and p, respectively, and remark briefly upon the singularities of these coefficients.

Throughout this paper Roman indices run from 0 to 3 and Greek indices from 0 to 2. We set $x^0 := t$, $x^1 := x$, $x^2 := y$, $x^3 := z$.

2. Reduction to three dimensions

Ω is defined by

$$\Omega(x^{k''}, x^{k'}) = \tfrac{1}{2}(\mu'' - \mu') \int_{\mu'}^{\mu''} g_{ij} \frac{dx^i}{d\mu} \frac{dx^j}{d\mu}\, d\mu,$$

where the integral is taken along the unique geodesic form $x^{k'}$ to $x^{k''}$, and μ is an affine parameter.

Observe that if $\gamma(\mu)$ is a geodesic in the Gödel V_4 and μ is an affine parameter, then the projection of γ on the first three coordinates is a geodesic in the V_3 whose metric is given by (1.2) and μ is also an affine parameter in this V_3. Hence

$$\begin{aligned}\Omega(x^{k''}, x^{k'}) &= \tfrac{1}{2}(\mu'' - \mu') \int_{\mu'}^{\mu''} g_{ij} \frac{dx^i}{d\mu} \frac{dx^j}{d\mu}\, d\mu \\ &= \tfrac{1}{2}(\mu'' - \mu') \int_{\mu'}^{\mu''} g_{\alpha\beta} \frac{dx^\alpha}{d\mu} \frac{dx^\beta}{d\mu}\, d\mu + \tfrac{1}{2}(\mu'' - \mu') \int_{\mu'}^{\mu''} \left(\frac{dz}{d\mu}\right)^2 d\mu \\ &= \Omega_1(x^{\alpha''}, x^{\alpha'}) + \tfrac{1}{2}(\mu'' - \mu') \int_{\mu'}^{\mu''} \left(\frac{dz}{d\mu}\right)^2 d\mu.\end{aligned}$$

Now $\xi^k := (0, 0, 0, 1)$ is a Killing field and hence $dz/d\mu = \text{constant}$ along the geodesic. Thus

$$dz/d\mu = (z'' - z')/(\mu'' - \mu'),$$

and so

$$\Omega(x^{k''}, x^{k'}) = \Omega_1(x^{\alpha''}, x^{\alpha'}) + \tfrac{1}{2}(z'' - z')^2. \tag{2.1}$$

Therefore it suffices to calculate Ω_1.

3. Reduction of the number of variables: reduced variables

If ξ^k is a Killing field in a V_4 then $\Omega(x^{k'}, x^k)$ must satisfy the equation

$$\xi^{k'}\Omega_{,k'}+\xi^k\Omega_{,k}=0.$$

For the Gödel metric there are five Killing fields:

$$\xi^k_{(1)}=(0,0,0,1), \qquad \xi^k_{(2)}=(0,0,1,0),$$
$$\xi^k_{(3)}=(1,0,0,0), \qquad \xi^k_{(4)}=(0,1,-\sqrt{2}\omega y, 0),$$
$$\xi^k_{(5)}=(-2\,\mathrm{e}^{-\sqrt{2}\omega x}, \sqrt{2}\omega y, \mathrm{e}^{-2\sqrt{2}\omega x}-\omega^2y^2, 0).$$

$\xi_{(1)}$, $\xi_{(2)}$, $\xi_{(3)}$ and $\xi_{(4)}$ are commonly quoted (e.g. Synge 1960, p 337) and $\xi_{(5)}$ can easily be shown to satisfy the Killing equations. This implies that the metric (1.2) has four Killing fields which are merely the projections of $\xi_{(2)}$, $\xi_{(3)}$, $\xi_{(4)}$ and $\xi_{(5)}$ onto the first three coordinates. Hence Ω_1 satisfies

$$\partial\Omega_1/\partial y+\partial\Omega_1/\partial y'=0, \tag{3.1}$$

$$\partial\Omega_1/\partial t+\partial\Omega_1/\partial t'=0, \tag{3.2}$$

$$\partial\Omega_1/\partial x-\sqrt{2}\omega y\;\partial\Omega_1/\partial y+\partial\Omega_1/\partial x'-\sqrt{2}\omega y'\;\partial\Omega/\partial y'=0, \tag{3.3}$$

$$\sqrt{2}\omega y\frac{\partial\Omega_1}{\partial x}+(\mathrm{e}^{-2\sqrt{2}\omega x}-\omega^2y^2)\frac{\partial\Omega_1}{\partial y}-2\,\mathrm{e}^{-\sqrt{2}\omega x}\frac{\partial\Omega_1}{\partial t}+\sqrt{2}\omega y'\frac{\partial\Omega_1}{\partial x'}$$
$$+(\mathrm{e}^{-2\sqrt{2}\omega x'}-\omega^2y'^2)\frac{\partial\Omega_1}{\partial y'}-2\,\mathrm{e}^{-\sqrt{2}\omega x'}\frac{\partial\Omega_1}{\partial t'}=0. \tag{3.4}$$

These equations may be solved by using the method outlined in Forsyth (1954). The result is that Ω_1 must be solely a function of the 'reduced variables'

$$p:=[(u'-u)^2+\omega^2\xi^2]/4uu' \tag{3.5}$$

and

$$q:=\omega\tau+\sin^{-1}\left(\frac{2(u+u')\omega\xi}{(u+u')^2+\omega^2\xi^2}\right), \tag{3.6}$$

where

$$u':=\mathrm{e}^{-\sqrt{2}\omega x'}, \qquad u:=\mathrm{e}^{-\sqrt{2}\omega x},$$
$$\xi:=y'-y, \qquad \tau:=t'-t.$$

The world function Ω_1 also satisfies the differential equations (Synge 1960, p 51)

$$g^{\alpha'\beta'}\Omega_{1,\alpha'}\Omega_{1,\beta'}=2\Omega_1, \qquad g^{\alpha\beta}\Omega_{1,\alpha}\Omega_{1,\beta}=2\Omega_1.$$

Explicitly,

$$\left(\frac{\partial\Omega_1}{\partial x'}\right)^2+2u'^2\left(\frac{\partial\Omega_1}{\partial y'}\right)^2-4u'\left(\frac{\partial\Omega_1}{\partial y'}\right)\left(\frac{\partial\Omega_1}{\partial t'}\right)+\left(\frac{\partial\Omega_1}{\partial t'}\right)^2=2\Omega_1,$$

and an identical equation with unprimed replacing primed variables. Using the fact that

Ω_1 depends only upon p and q, both these equations reduce to the same, formally very simple, equation

$$2p(p+1)^2\left(\frac{\partial\Omega_1}{\partial p}\right)^2+(p-1)\left(\frac{\partial\Omega_1}{\partial q}\right)^2=\frac{2}{\omega^2}(p+1)\Omega_1. \tag{3.7}$$

4. Ω in parametric form

In equation (3.7) set $\Omega_1 := \frac{1}{2}\epsilon V_1^2$, where ϵ is a constant which takes the values $-1, 0, 1$ for time-like, light-like and space-like geodesics respectively, in the V_3 whose metric is given by (1.2). Then

$$2p(p+1)^2(\partial V_1/\partial p)^2+(p-1)(\partial V_1/\partial q)^2=\epsilon\omega^{-2}(p+1). \tag{4.1}$$

The usual prescription for solving the Hamilton–Jacobi equations for the characteristic function (Conway and McConnell 1940, editors' appendix, note 2) may be generalised to cover also reduced Hamilton–Jacobi equations; see Warner (1978, § 9). In the extreme case (4.1) where only a single equation remains the prescription is this: find a complete integral $f(p, q; k)$ of the equation (4.1) with f replacing V_1, and then

$$V_1(p,q)=f(p,q;k), \tag{4.2}$$

where k is to be eliminated in favour of p, q by means of the condition

$$\partial f/\partial k=0. \tag{4.3}$$

Since (4.1) is separable the determination of f is straightforward:

$$V_1=\omega^{-1}\left[\left(\frac{\epsilon}{1+\lambda}\right)^{1/2}\int\frac{\mathrm{d}p(p+\lambda p^2)^{1/2}}{p(p+1)}+kq\right], \tag{4.4}$$

with

$$\lambda\ \iota\ (\epsilon-k^2)/(\epsilon+k^2). \tag{4.5}$$

(4.3) now reads explicitly

$$\frac{(1-\lambda)^{1/2}}{2}\int\frac{(1-p)\,\mathrm{d}p}{(1+p)(p+\lambda p^2)^{1/2}}+q=0. \tag{4.6}$$

(As a matter of convenience it has been assumed temporarily that $0<\lambda<1$ here.) q may now be removed from (4.4) by means of (4.6) to give

$$V_1=\tfrac{1}{2}[\epsilon(\lambda+1)]^{1/2}\int(p+\lambda p^2)^{-1/2}\,\mathrm{d}p. \tag{4.7}$$

The various integrals above are elementary. Before evaluating them one should note the following restrictions on the values of λ: (i) $-\infty<\lambda<-1$ when $\epsilon=-1$ and (ii) $-1<\lambda<1$ when $\epsilon=1$, the light-like case corresponding to $\lambda=-1$. When $\lambda<0$ we set $\lambda' := -\lambda$. Then explicitly

(i) $\epsilon=-1$, $1<\lambda'<\infty$,

$$V_1=(1/\omega)[(\lambda'-1)/\lambda']^{1/2}\sin^{-1}(\lambda' p)^{1/2}, \tag{4.8a}$$

$$0=[(\lambda'+1)/\lambda']^{1/2}\sin^{-1}(\lambda' p)^{1/2}-2\sin^{-1}[(\lambda'+1)p/(p+1)]^{1/2}-q; \tag{4.8b}$$

(iia) $\epsilon = 1$, $0 < \lambda' < 1$,

$$V_1 = (1/\omega)[(1-\lambda')/\lambda']^{1/2} \sin^{-1}(\lambda' p)^{1/2}, \tag{4.9}$$

together with (4.8b);

(iib) $\epsilon = 1$, $0 < \lambda < 1$,

$$V_1 = (1/\omega)[(1+\lambda)/\lambda]^{1/2} \sinh^{-1}(\lambda p)^{1/2}, \tag{4.10a}$$

$$0 = (1-\lambda/\lambda)^{1/2} \sinh^{-1}(\lambda p)^{1/2} - 2\sin^{-1}[(1-\lambda)p/(p+1)]^{1/2} - q. \tag{4.10b}$$

In the time-like case, for example, the world function Ω is then given by

$$\Omega = -[(\lambda'-1)/2\lambda'\omega^2][\sin^{-1}(\lambda' p)^{1/2}]^2 + \tfrac{1}{2}(z'-z)^2, \tag{4.11}$$

with λ' given by (4.8b). It may be confirmed that in the limit $\omega \to 0$ one obtains from this the correct flat-space world function

$$\Omega_{\omega=0} = -\tfrac{1}{2}(t'-t)^2 + \tfrac{1}{2}(x'-x)^2 - \tfrac{1}{4}(y'-y)^2 + \tfrac{1}{2}(z'-z)^2 - (t'-t)(y'-y). \tag{4.12}$$

It remains to consider the light-like case. Then $\lambda' = 1$ so that $V_1 = 0$, of course. The equation of the light cone is given by (4.8b), namely

$$q = \sqrt{2}\sin^{-1} p^{1/2} - 2\sin^{-1}[2p/(p+1)]^{1/2}. \tag{4.13}$$

It should be borne in mind that, except in the case of equation (4.11), we have been dealing with the subspace $z = \text{constant}$. Therefore (4.13), for instance, is really the equation of the intersection of the light cone with planes $z = \text{constant}$; and in the 'time-like case' and the 'light-like case' the geodesics in the original V_4 may of course be space-like: for this to occur it is only necessary for $(z'-z)^2$ to be sufficiently large.

5. Series expansion and analyticity

Since the metric (1.2) is analytic in x, y and t, it follows (Buchdahl and Warner 1979) that Ω_1 is an analytic function of x^α and $x^{\alpha'}$ for $|x^{\alpha'} - x^\alpha|$ sufficiently small. We wish to show that Ω_1 is analytic in p and q, for p and q sufficiently small.

Expand Ω_1 into a series in ascending powers of x', y', t', x, y, t. Equations (3.1) and (3.2) imply that the terms in y', y, t', t can be grouped into terms involving only $\xi := y'-y$ and $\tau := t'-t$. Observe that the metric is invariant under simultaneous sign reversal of y and t. Hence Ω_1 must be invariant under simultaneous interchange of y' with y and t' with t. This means that ξ and τ can appear in the series only in the combinations ξ^2, $\xi\tau$ and τ^2. Define

$$\eta_1 := x',\ \eta_2 := x,\ \eta_3 := \xi^2,\ \eta_4 := \xi\tau,\ \eta_5 := \tau^2$$

and

$$z_1 := x',\ z_2 := x,\ z_3 := p,\ z_4 := q^2,\ z_5 := \xi\tau.$$

From the preceding argument, Ω_1 is analytic in the variables $\eta_1, \ldots, \eta_5$.

Let

$$f(\xi) := \xi^{-1} \sin^{-1}\left(\frac{2(u+u')\omega\xi}{(u+u')^2 + \omega^2\xi^2}\right).$$

Observe that f is analytic in ξ, x, x' provided $|\omega\xi| < u' + u$. Note also that only even powers of ξ appear in a power series expansion of f about $\xi = 0$. There therefore exists an analytic function g such that $g(\eta_3) = f(\xi)$. Now

$$\begin{aligned} &z_1 = \eta_1, \qquad z_2 = \eta_2, \\ &z_3 = \tfrac{1}{4}\exp[\sqrt{2}\omega(\eta_1+\eta_2)]\{[\exp(-\sqrt{2}\omega\eta_1) - \exp(-\sqrt{2}\omega\eta_2)]^2 + \omega^2\eta_3\}, \\ &z_4 = \omega^2\eta_5 + 2\omega\eta_4 g(\eta_3) + \eta_3 g^2(\eta_3), \qquad z_5 = \eta_4, \end{aligned}$$

so that $z_1, \ldots, z_5$ are analytic functions of $\eta_1, \ldots, \eta_5$. Furthermore the Jacobian determinant of this transformation is simply

$$-\frac{\partial z_3}{\partial \eta_3}\frac{\partial z_4}{\partial \eta_5} = -\frac{\omega^4}{4}\exp[\sqrt{2}\omega(\eta_1+\eta_2)],$$

which vanishes nowhere. Hence locally we can analytically invert this transformation. When $|x^{\alpha'} - x^{\alpha}|$ is sufficiently small, Ω_1 is therefore an analytic function of $z_1, \ldots, z_5$ and thus of p and q^2.

The power series for Ω_1 has the generic form

$$\begin{aligned} \Omega_1 = k &+ (a_1 p + a_2 q^2) + (b_1 p^2 + b_2 p q^2 + b_3 q^4) + (c_1 p^3 + c_2 p^2 q^2 + c_3 p q^4 + c_4 q^6) \\ &+ (d_1 p^4 + d_2 p^3 q^2 + d_3 p^2 q^4 + d_4 p q^6 + d_5 q^8) + \ldots. \end{aligned} \tag{5.1}$$

Employing the coincidence limits of Ω, $\Omega_{;i}$ and $\Omega_{;ij}$ (Synge 1960, p 57), one can show that $k = 0$, $a_1 = \omega^{-2}$ and $a_2 = -\frac{1}{2}\omega^{-2}$. The remaining terms can be found by substitution of (5.1) into (3.7). The result is

$$\begin{aligned} \Omega_1 = \omega^{-2}[&(p - \tfrac{1}{2}q^2) - \tfrac{1}{3}(p^2 + pq^2) + \tfrac{1}{45}(8p^3 + p^2q^2 - pq^4) \\ &- \tfrac{1}{945}(108p^4 + 23p^3q^2 + 25p^2q^4 + 2pq^6)] + O_5, \end{aligned} \tag{5.2}$$

where O_5 denotes terms of degree 5 or more in p and q^2.

It is not easy to find useful limits on the values of p and q which ensure convergence of (5.2). The preceding analyticity arguments will give some limits, but these are exceedingly weak.

6. Power series in p

We now consider a power series of the form

$$\Omega_1 = a_0(q) + a_1(q)p + a_2(q)p^2 + a_3(q)p^3 + O(p^4). \tag{6.1}$$

Substitution of this into (3.7) yields the differential equations

$$\begin{aligned} &\dot{a}_0^2 + 2\omega^{-2}a_0 = 0, \\ &\dot{a}_0\dot{a}_1 - a_1^2 - \tfrac{1}{2}\dot{a}_0^2 + \omega^{-2}(a_1 + a_0) = 0, \\ &\dot{a}_0\dot{a}_2 - 4a_1a_2 + \tfrac{1}{2}\dot{a}_1^2 - \dot{a}_0\dot{a}_1 - 2a_1^2 + \omega^{-2}(a_2 + a_1) = 0, \\ &\dot{a}_0\dot{a}_3 - 6a_1a_3 + \dot{a}_1\dot{a}_2 - \dot{a}_0\dot{a}_2 - \tfrac{1}{2}\dot{a}_1^2 - 4a_2^2 - 8a_1a_2 - a_1^2 + \omega^{-2}(a_3 + a_2) = 0, \end{aligned}$$

dots indicating differentiation with respect to q. These can be integrated to give

$$\begin{aligned}
a_0 &= -\tfrac{1}{2}\omega^{-2}q^2,\\
a_1 &= \omega^{-2}q \cot q,\\
a_2 &= -\tfrac{1}{4}\omega^{-2}[q^2 \operatorname{cosec}^4 q + q \cot q(2-3 \operatorname{cosec}^2 q) + 2\cot^2 q],\\
a_3 &= \tfrac{1}{24}\omega^{-2}[6q^3 \cot q \operatorname{cosec}^6 q + 3q^2 \operatorname{cosec}^4 q(8-9\operatorname{cosec}^2 q)\\
&\quad + q \cot q(8-32\operatorname{cosec}^2 q + 39 \operatorname{cosec}^4 q) + 6\cot^2 q(2-3\operatorname{cosec}^2 q)].
\end{aligned} \tag{6.2}$$

The choice of the constants of integration is made in such a way that the a_i are even functions of q, not singular as $q \to 0$, and so that they are consistent with the coincidence limits of Ω, $\Omega_{;i}$ and $\Omega_{;ij}$. One can easily check that (6.1) and (6.2) are in harmony with (5.2).

It is interesting to note that as $q \to \pm\pi$, a_1, a_2 and a_3 become singular. The limit $q \to \pm\pi$ becomes $t'-t \to \pm\pi/\omega$ as $\xi \to 0$. However, Kundt (1956) has shown that for geodesics in an ϵ neighbourhood of the t axis, the points (x, y, t') and (x, y, t) are conjugate for

$$t'-t = k\pi/\omega + O(\epsilon), \qquad k = \pm 1, \pm 2, \ldots.$$

Therefore the singular points of the coefficients of the series (6.1) occur at the nearest pair of conjugate points on such geodesics in the limit $p \to 0$ along their length.

7. Power series in q

This time we look for a power series solution of the form

$$\Omega_1 = b_0(p) + b_1(p)q^2 + b_2(p)q^4 + O(q^6). \tag{7.1}$$

Substitution of this into (3.7) yields the equations

$$\begin{aligned}
&p(p+1)\dot b_0^2 - \omega^{-2}b_0 = 0,\\
&2p(p+1)^2\dot b_0\dot b_1 + 2(p-1)b_1^2 - \omega^{-2}(p+1)b_1 = 0,\\
&p(p+1)^2(\dot b_1^2 + 2\dot b_0\dot b_2) + 8(p-1)b_1b_2 - \omega^{-2}(p+1)b_2 = 0.
\end{aligned}$$

The equations are simplified by the change of variable $p =: \sinh^2 r$. If dots now indicate differentiation with respect to r,

$$\begin{aligned}
&\dot b_0^2 - 4\omega^{-2}b_0 = 0,\\
&\dot b_0\dot b_1 + 4(\tanh^2 r - \operatorname{sech}^2 r)b_1^2 - 2\omega^{-2}b_1 = 0,\\
&\dot b_0\dot b_2 + \tfrac{1}{2}\dot b_1^2 + 16(\tanh^2 r - \operatorname{sech}^2 r)b_1b_2 - 2\omega^{-2}b_2 = 0.
\end{aligned}$$

These are easily integrated to give

$$\begin{aligned}
&b_0 = r^2/\omega^2, \qquad b_1 = r/2\omega^2(r - 2\tanh r),\\
&b_2 = \frac{1}{12\omega^2}\,\frac{\tanh r}{(r-2\tanh r)^4}[r(\tanh^2 r - 3) + 3\tanh r].
\end{aligned} \tag{7.2}$$

The constants of integration are again determined in such a way that the b_α are not singular as $p \to 0$, and so that they are consistent with the coincidence limits of Ω, $\Omega_{;i}$ and $\Omega_{;ii'}$. Once again, (7.1) and (7.2) are in harmony with (5.2).

It is apparent that b_1 and b_2 are singular at r_0, where r_0 satisfies $r_0 = 2 \tanh r_0$, $r_0 = 1{\cdot}915\ldots$. We conjecture that this corresponds to the appearance of pairs of points which can be connected by more than one geodesic when $r \geqslant r_0$.

8. Concluding remarks

In the course of this paper we have solved the partial differential equation (3.7) in power series without having encountered any problems with the determination of arbitrary functions. The selection of the appropriate solution of (3.7) has been achieved (i) by appealing to the analyticity of Ω in the capacity of a function of suitably chosen variables, and (ii) by being able to specify the actual or generic form of the initial terms of such a series by appealing to coincidence limits.

In our experience with world functions we have found that singularities seem to appear for one of two reasons. First, we shall obviously have singular behaviour if the metric itself is singular or degenerate at some point, such as in the case of the origin of the Schwarzschild metric (Buchdahl and Warner 1979). Secondly, the world function will fail to be analytic in a region if it contains a pair of points which can be connected by more than one geodesic. This is not surprising when one bears in mind that the very definition of $\Omega(x^{k''}, x^{k'})$ requires the geodesic from $x^{k'}$ to $x^{k''}$ to be unique.

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